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Threshold Functions and Bounded Depth Monotone Circuits

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We prove an exponential lower bound for the majority function on constant depth monotone circuits, solving an open problem posed by Yao (*in* "Proceedings of 24th IEEE Sympos. Found. of Comput. Sci.," Tucson, 1983, pp. 420–428). In particular, we prove that computing majority on depth d monotone circuits requires $\exp \Omega(n^{1/(d-1)})$ size. This result implies exponential lower bounds for other functions, such as testing connectivity and detecting cliques in graphs.

1. INTRODUCTION

Bounded depth circuits are Boolean circuits with a bounded number of alternating levels of AND gates and OR gates each having unbounded fan-in and fan-out. The inputs are Boolean variables and their negations. A circuit is said to be monotone if it does not have negated variables as inputs. The size of a circuit is the number of gates it contains, and its depth is the number of alternating levels. A Σ_d circuit is a depth d circuit whose top level (the level farthest from the inputs) contains an OR gate. A Π_d circuit is a depth d circuit whose top level contains an AND gate.

Many papers have appeared recently about bounded depth circuits. Furst, Saxe, and Sipser [7] showed that both the parity function (which is 1 if an odd number of its variables are 1) and the majority function (which is 1 if at least one-half of its variables are 1) both require more than polynomial size to compute on constant depth circuits. The methods of Ajtai [1] improve the lower bound to $\exp \Omega((\log n)^2)$ size (the constant depends on the depth). Even this result is not close to the best known upper bound, which is $\exp O(n^{1/(d-1)})$ size, where d is the depth of the circuit.

Better lower bounds have been proven in the weaker model of monotone circuits. Valiant [11] gave an exponential lower bound for monotone Σ_3 circuits which detect cliques in graphs. Yao [13] subsequently gave an exponential lower bound

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for depth 3 monotone circuits computing the majority function, but he leaves as an open problem whether exponential lower bounds exist for depths higher than 3.

In this paper we show that majority requires exponential size for a monotone circuit of *any* constant depth. Specifically, we prove that the majority function on depth d monotone circuits requires $\exp \Omega(n^{1/(d-1)})$ size to compute. This lower bound gives a depth-size trade-off. For example, we can show that computing majority on polynomial size monotone circuits requires $\Omega(\log n/\log \log n)$ depth, and we also show that this bound is tight.

These results imply exponential lower bounds for other problems as well. The majority function is reducible via monotone projections (substitutions of constants and unnegated variables [10]) to many graph problems: testing connectivity, detecting cliques, and detecting Hamiltonian cycles. We thus get exponential lower bounds in our model for these problems. A threshold function is a function which is 1 if at least t of the variables are 1 for some given t (majority is the special case $t = \frac{1}{2}n$). As long as $n^{\varepsilon} \leqslant t \leqslant n - n^{\varepsilon}$ for some $\varepsilon > 0$, we can prove that majority is a monotone projection of the threshold-t function, which means that threshold-t requires exponential size on a constant depth monotone circuit.

Exponential lower bounds for monotone circuits have also been obtained by Klawe *et al.* [8]. They exhibit functions having monotone Σ_d circuits of linear size, but which require $\exp \Omega(n^{1/(d-1)})$ size on monotone Π_d circuits. Sipser [9] had shown a superpolynomial lower bound for the same problem without the monotone restriction.

The remainder of this paper is divided into four sections. Section 2 gives the lower bound for the majority function. Projections to other functions are presented in Section 3. Section 4 contains upper bounds for threshold functions. Finally in Section 5 we conclude with some open problems.

2. Lower Bounds

The lower bound technique used by both Valiant and Yao involves showing that a small depth 2 subcircuit of a depth 3 circuit can only pick up an exponentially small fraction of the prime implicants of the function in question. Either the depth 2 subcircuits are large or a large number of them are required, so in either case large size is required for the entire depth 3 circuit. This idea does not seem to extend directly for higher depths.

Our approach is similar to that of [7]: we reduce a depth d circuit to a depth $d-1$ circuit by performing a restriction, that is, assigning some of the variables values, but still keeping it a majority function. Here we take advantage of monotonicity to do the reduction for larger sizes than the method of [7] could handle. The reduction from depth d to depth $d-1$ is done by switching the order of the gate types of the bottom two levels (see Lemma 2.1), and then collapsing the two adjacent levels with the same gate type into one level. Here the bottom level means the level closest to the input.

The lemma below shows how to convert Π_2 circuits to Σ_2 circuits, and vice versa.

LEMMA 2.1. *Let F be a monotone Boolean function with n inputs and m outputs. Suppose that F is computed by a Π_2 monotone circuit C whose bottom level fan-in is bounded by k. If the condition $(n/(n'k))^l \geqslant m$ is satisfied, then there is a restriction of F to n' inputs (the other $n-n'$ inputs are set to 0) which is computed by a Σ_2 monotone circuit C' with bottom level fan-in bounded by l (we can also convert Σ_2 circuits to Π_2 circuits using a dual argument).*

Proof. An output of C will have the form $C_1 \wedge \cdots \wedge C_j$, where the C_i are disjuncts of at most k variables. For this output of C to be 1, at least one of the variables in C_1 must be 1. There are at most k choices of a variable from C_1. For each choice, throw out the C_i which contain that variable and look for a remaining disjunct. There are at most k choices of a variable for this disjunct as well. Continuing in this manner, we obtain a tree of possible choices to turn the output of C on. This tree has a branching factor of at most k.

Let us stop the tree when it reaches a depth of l. The function computed by the tree will change after this action, but it is "close" to the original function in a sense which will be defined below. Then by $\bigwedge$-ing all the variables in a particular path, and then $\bigvee$-ing all of these together, we will have a Σ_2 monotone circuit with bottom level fan-in l. In a single tree at most k^l paths were prematurely stopped, because the tree has depth at most l and branching factor at most k. There is one tree for each of the m outputs of C, so there are at most mk^l stopped paths in all.

The idea now is to choose $n-n'$ of the n variables which intersect all of the at most mk^l stopped paths. By setting these variables to 0, we can ignore the stopped paths because they are now 0 as well. So under the restriction C will change to a Σ_2 monotone circuit with bottom level fan-in bounded by l, and we would be done.

A probabilistic argument shows that such a restriction exists. Choose $n-n'$ of the n variables randomly. Then the probability that the choice does not intersect all stopped paths is at most mk^l times the probability that the choice does not intersect a particular path of length l, which equals

$$\begin{aligned} mk^l \frac{\binom{n-l}{n-n'}}{\binom{n}{n-n'}} &= mk^l \frac{\binom{n'}{l}}{\binom{n}{l}} \\ &= mk^l \frac{(n')^l}{n^l} \frac{1\cdot(1-1/n')\cdots(1-(l-1)/n')}{1\cdot(1-1/n)\cdots(1-(l-1)/n)} \\ &< m\left(\frac{kn'}{n}\right)^l \\ &\leqslant 1. \end{aligned}$$

Thus the probability that the choice does intersect all stopped paths is greater than zero, which means that some choice exists which intersects all stopped paths. Setting these variables to 0, we are done. ∎

DEFINITION. Let $P(d, n, s, k)$ be the property that there exists a depth d monotone circuit C computing the majority function of n variables such that

(1) the number of gates in C excluding the bottom level is at most s, and

(2) the bottom level fan-in is at most k.

The lemma below shows how to reduce depth by one.

LEMMA 2.2. *If* $d \geqslant 3$ and $(n/(2n'k))^l \geqslant s$, *then* $P(d, n, s, k)$ *implies* $P(d-1, n', s, l)$.

Proof. Let C be a circuit satisfying the property $P(d, n, s, k)$ and assume that the bottom level of C is composed of $\bigvee$-gates (if the bottom level is composed of $\bigwedge$-gates, the dual of the argument will work). Set the first $\frac{1}{2}(n-n')$ variables to 1. There are at most s $\bigwedge$-gates on the second level from the bottom. Since $(\frac{1}{2}(n+n')/(n'k))^l > (n/(2n'k))^l \geqslant s$, we may apply Lemma 2.1 to convert the bottom two levels into Σ_2 form with bottom level fan-in bounded by l. By collapsing the two adjacent $\bigvee$-levels into one level, we reduce the depth by one. The new function computed is the majority function of n' variables. Thus the predicate $P(d, n', s, l)$ is satisfied, and we are done. ∎

THEOREM 2.3. *The majority function on n variables requires* $\exp \Omega(n^{1/(d-1)})$ size on a depth d monotone circuit.

Proof. Let C be a depth d monotone circuit of size s computing majority. Thinking of C as a depth $d+1$ circuit with bottom level fan-in 1, we see that the property $P(d+1, n, s, 1)$ is true. Choose $n_d = n/6$ (rounded to nearest integer). Observing that $(n/(2n_d))^{\log s} = 3^{\log s} > s$ (where the logarithm is base e), we may apply Lemma 2.2 to get the property $P(d, n_d, s, \log s)$. Now define $n_{i-1} = n_i/(6 \log s)$ for $i = d, d-1, ..., 3$. Since $(n_i/(2n_{i-1} \log s))^{\log s} = 3^{\log s} > s$, we may apply Lemma 2.2 $d-2$ times iteratively to find that the predicate $P(2, n_2, s, \log s)$ is true, where $n_2 = n_d/(6 \log s)^{d-2} = n/(6^{d-1}(\log s)^{d-2})$.

But depth 2 circuits computing majority must have fan-in at least $\frac{1}{2}$ the number of variables. Thus we have $\log s \geqslant \frac{1}{2}n_2 \geqslant \frac{1}{2}n/(6^{d-1}(\log s)^{d-2})$, which implies $(6 \log s)^{d-1} \geqslant \frac{1}{2}n$, so $s = \exp \Omega(n^{1/(d-1)})$. ∎

COROLLARY 2.4. *Majority on polynomial size monotone circuits requires* $\Omega(\log n/\log \log n)$ *depth.*

Proof. If s is polynomial, we have $\log s = O(\log n)$, which together with Therem 2.3 implies the corollary. ∎

3. Projections to Other Functions

In this section, we will show that majority is a monotone projection of other functions: testing connectivity, detecting cliques, and most threshold-t functions. As a corollary, we get exponential lower bounds for these functions.

Definition (see [10]). $F=\{f_n\}$ is said to be a *monotone polynomial projection* of $G=\{g_n\}$ (written $F \leqslant_{m-\text{proj}} G$) if there is a function $p(n)$ bounded by a polynomial in n, and for each n there exist literals $y_1,\ldots,y_{p(n)} \in \{x_1,\ldots,x_n,0,1\}$ such that

$$f_n(x_1,\ldots,x_n)=g_{p(n)}(y_1,\ldots,y_{p(n)}).$$

Note that if $F \leqslant_{m-\text{proj}} G$ and F requires exponential size on constant depth monotone circuits, then so does G (with perhaps a different exponent). Also if F requires depth $d(n)$ to compute in polynomial size, then so does G.

Consider directed graphs with vertices $\{1,\ldots,n\}$. For $1 \leqslant i, j \leqslant n$, define x_{ij} to be 1 if and only if there is an edge from i to j (we allow edges from a vertex to itself). We define the following graph predicates.

Definitions. (1) *connectivity*$_n$: Given $\{x_{ij}\}$, is there a directed path from 1 to n?

(2) *clique*$_n$: Given $\{x_{ij}\}$, is there a complete subgraph with a least $\frac{1}{2}n$ vertices?

Theorem 3.1. *majority* $\leqslant_{m-\text{proj}}$ *connectivity.*

Proof. Skyum and Valiant [10] have shown a more general result, namely that any function with monotone polynomial size formulas is a monotone projection of connectivity. Since Valiant [12] has shown that majority has polynomial size monotone formulas, the proof is complete.

There is a more direct and compact projection (shown to me by Michael Sipser). Let the set of vertices be the triangular grid $\{(i,j): 0 \leqslant i \leqslant j \leqslant n\}$. The edge from (i_1,j_1) to (i_2,j_2) is labelled as follows:

(1) if $i_2=i_1$ and $j_2=j_1+1$, then the edge is labelled 1;

(2) if $i_2=i_1+1$ and $j_2=j_1+1$, then the edge is labelled x_{j_2};

(3) otherwise, the edge is labelled 0.

In this graph, there is a path from vertex $(0,0)$ to vertex (t,n) if and only if at least t of the x_i are 1. By setting t to $\frac{1}{2}n$ we are finished. ▌

Theorem 3.2. *majority* $\leqslant_{m-\text{proj}}$ *clique.*

Proof. Given variables $\{x_1,\ldots,x_n\}$, we will construct a directed graph which has a complete subgraph of t vertices if and only if t of the x_i are 1. By setting t equal to $\frac{1}{2}n$ we will be finished.

Let the vertices of the graph be $\{1, ..., n\}$, and define the edge y_{ij} as follows: if $i=j$ then let $y_{ij}=x_i$, otherwise let $y_{ij}=1$. Then t particular variables will all be 1 if and only if the corresponding subgraph is complete. Thus there is a complete subgraph of at least $\frac{1}{2}n$ vertices if and only if the majority of the x_i are 1 (our proof uses the fact that edges are allowed between a vertex and itself, but actually projections can be given which don't contain such edges). ∎

THEOREM 3.3. *If $n^{\varepsilon} \leqslant t(n) \leqslant n-n^{\varepsilon}$ for some $\varepsilon>0$, then majority $\leqslant_{m-\text{proj}}$ threshold-t.*

Proof. Given variables $\{x_1, ..., x_n\}$, we will show how to compute the majority of the variables by computing the threshold-t function of some other values. Set $m=\lceil (n/2)^{1/\varepsilon} \rceil$, $p=t(m)-\frac{1}{2}n$, and $q=m-t(m)-\frac{1}{2}n$. Observe that $p=t(m)-\frac{1}{2}n \geqslant m^{\varepsilon}-\frac{1}{2}n \geqslant 0$, and $q=m-t(m)-\frac{1}{2}n \geqslant m^{\varepsilon}-\frac{1}{2}n \geqslant 0$. We will now compute the threshold-t function on the values x_1 through x_n, plus p1's and q0's, for a total of $n+p+q=m$ values. This threshold-t computation will be 1 if at least $t(m)$ of the values are 1, which means that at least $t(m)-p$ of the x_i are 1. Since $t(m)-p$ equals $\frac{1}{2}n$, we have computed majority. ∎

COROLLARY 3.4. *Connectivity and clique require exponential size to compute with constant depth monotone circuits. The same is true for threshold-t provided that $n^{\varepsilon} \leqslant t(n) \leqslant n-n^{\varepsilon}$ for some $\varepsilon>0$.*

Proof. Follows from Theorems 3.1, 3.2, and 3.3. ∎

4. UPPER BOUNDS

In this section we will provide upper bounds for majority and other threshold functions. Theorem 4.1 shows that the lower bound of Theorem 2.3 is tight up to logarithmic factors, while Theorem 4.2 proves that Corollary 2.4 is tight. Theorem 4.3 shows that certain threshold functions can be defined in constant depth with polynomial size monotone circuits, and Theorem 4.4 shows that Theorem 4.3 is best possible.

THEOREM 4.1. *For $d<\log n$, majority can be computed with depth d monotone circuits using $\exp O(n^{1/(d-1)}(\log n)^{(d-2)/(d-1)})$ size.*

Proof. We will in fact construct all of the threshold functions of n variables simultaneously in the required size. We will break up the variables into a_1 parts, each broken into a_2 parts, each broken into a_{d-1} parts, where $n=a_1 \cdots a_{d-1}$. Each part will be done in depth 2, and by doing them alternately in Σ_2 and Π_2, we will end up with a depth d circuit. The bottom part can be done in size $n \cdot 2^{a_{d-1}}$, by putting it into conjunctive normal form. The ith part (for $i<d-1$) can be done in size $n \cdot n^{a_i}$ by combining all possibilities for the threshold functions of the part below

it. The total size is thus bounded by $n \cdot (n^{a_1} + \cdots + n^{a_{d-2}} + 2^{a_{d-1}})$. Now let $a_i = (n/\log n)^{1/(d-1)}$ for $i < d-1$ and let $a_{d-1} = n^{1/(d-1)}(\log n)^{(d-2)/(d-1)}$. The size then becomes $\exp O(n^{1/(d-1)}(\log n)^{(d-2)/(d-1)})$. ∎

THEOREM 4.2. *Majority has polynomial size monotone circuits of depth* $O(\log n/\log\log n)$.

Proof. Valiant [12] shows that majority has polynomial size, $O(\log n)$ depth montone formulas, where fan-in is bounded by 2. Chandra, Stockmeyer, and Vishkin [4, Theorem 3.4] show how to convert such formulas into circuits of depth $O(\log n/\log\log n)$ using unbounded fan-in and polynomial size. Since their method preserves monotonicity, we are done. ∎

We mention another upper bound for threshold functions, proven by [2, 5, and 6].

THEOREM 4.3. *If for some* $c > 0$ *we have* $\min(t(n), n - t(n)) = O((\log n)^c)$, *then threshold-t can be computed with constant depth polynomial size monotone circuits.*

Proof. Actually the above three papers prove the theorem without the monotone restriction. However, the method of [6] does not use negations, so we are done. In fact, the methods of [2 and 5] can be modified to give monotone circuits as well. ∎

We can prove a converse to the above theorem.

THEOREM 4.4. *If threshold-t can be computed with depth d polynomial size monotone circuits, then* $\min(t(n), n - t(n)) = O((\log n)^{d-1})$.

Proof. Assume that threshold-t had depth d polynomial size monotone circuits, where $\min(t(n), n - t(n))$ was not $O((\log n)^{d-1})$. Then by a proof similar to that of Theorem 3.3, we could construct depth d monotone circuits for majority that did not have $\exp \Omega(n^{1/(d-1)})$ size. But such circuits do not exist (by Theorem 2.3), so our assumption was false and the theorem is proved. ∎

5. CONCLUSIONS

We have shown that majority requires $\exp \Omega(n^{1/(d-1)})$ size using a depth d monotone circuit. This result implies that majority requires $\Omega(\log n/\log\log n)$ depth on a polynomial size monotone circuit. This bound is optimal, since majority can be done in $O(\log n/\log\log n)$ depth.

The situation for nonmonotone circuits is not as nice. Do the above lower bounds hold for nonmonotone circuits as well? This question is still open. Such lower bounds would imply results on the separation of the relativized polynomial-time hierarchy (see [7, 9]). Even exponential lower bounds for parity or majority in depth 3 have not been found.

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